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Federico Daniel Platania

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Banca, Finanzas y Actividad Empresarial

Editorial Universidad Cantabria

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Sector Financiero fruto de todas las acciones desarrolladas en el ámbito de la Fundación, y especialmente los resultados de las Becas, Ayudas y Premios a Tesis Doctorales.

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Finalmente destacar su participación como patrono en la creación, en alianza con las Universidades de Murcia, Politécnica de Cartagena y Cantabria, de la Fundación para el Análisis Estratégico y Desarrollo de la Pyme, en cuyo seno se crea la Red Internacional de Investigadores en Pymes. Fruto de esta actuación se elaboran diversos Informes sobre la Pyme en Iberoamérica, tanto a nivel de la región en su conjunto como en los distintos países.

FRANCISCO JAVIER MARTÍNEZ GARCÍA
Director de la Fundación UCEIF

ÍNDICE

INTRODUCTION

The stochastic behaviour of interest rates and commodity prices have been thoroughly analysed in the academic literature and constitutes an issue of special relevance for practitioners in financial markets. Previous studies have proposed numerous processes to model the stochastic component of these assets, most of them assuming a mean reverting process.

On the one hand, Table 1 presents some spot rate models proposed in the academic literature, classified in two categories: endogenous and exogenous. Endogenous models assume that changes in interest rates are affected by one or more factors and propose a certain stochastic behaviour for the factors. Under those assumptions, the current term structure can be derived as an implication from the model. Popular examples of one-factor models are Vasicek (1977), Brennan and Schwartz (1980), or Cox et al. (1985). The downside of these models is the lack of an appropriate fit to observed interest rate data. To mitigate this drawback some multi-factor models have been proposed. See, for instance, Brennan and Schwartz (1979), Schaefer and Schwartz (1984), Longstaff and Schwartz (1992), Duffie and Kan (1996), or Chen (1996). In contrast, exogenous models consider the current term structure as an input and aim to prevent arbitrage opportunities considering interest rates with different maturities. A pioneer work in this area was made by Ho and Lee (1986) who proposed a model consistent with observed data. As this model implies a Gaussian distribution and no mean reversion for interest rates, several papers have specified and analysed alternative model specifications such as Black et al. (1990), Hull and White (1990, 1993), Black and Karasinski (1991), Heath et al. (1992), and Mercurio and Moraleda (2000). For a complete survey on term structure models see, for instance, Webber and James (2001), Brigo and Mercurio (2006), or Filipovi´c (2009).

On the other hand, we can also find a significant number of papers addressing empirically and theoretically the commodity valuation problem. For instance, Schwartz (1997) compares three mean- reverting models for the stochastic behaviour of commodity, i) a simple one-factor model based on the logarithm of the commodity spot price, ii) a two-factor model proposed in Gibson and Schwartz (1990), where the second factor accounts for the convenience yield of the commodity, and iii) an extension of the Gibson and Schwartz (1990) model that incorporates the stochastic behaviour of interest rates as in Vasicek (1977). Schwartz and Smith (2000) present a representation of the two-factor model, where the log-spot price is described as the sum of two state variables referred to as the short-term deviation in prices and the equilibrium price level, respectively. Moreover, Lucia and Schwartz (2002) address the possible seasonal behaviour of the commodity price. In this paper the authors use the Scandinavian electricity market to compare a number of models based on the spot price and the logarithm of the spot price, where the seasonal component is arbitrary added in the spot (log-spot) price and modelled by a deterministic trigonometric function with annual frequency. On this regard, Cartea and Figueroa (2005) extend the one-factor model allowing the stochastic process to follow a zero level mean-reverting jump-diffusion process for the underlying log-spot price and the exponential of the trigonometric function is replaced by a Fourier series of order five. For a thorough description of some commodity models see, for instance, Pilipovi´c (1998).

In this work we extend the existing literature allowing the underlying state variable to capture any possible seasonal or cyclical behaviour. On this regard, section 2 analyses a continuous-time model for the term structure of interest rates where the spot rate is assumed to converge to a long- term level that changes over time according to a Fourier series. Section 3 proposes a square-root model where the instantaneous interest rate is pulled back to a certain time-dependent long term level characterized by an harmonic oscillator. Section 4 introduces a continuous-time model based on an Ornstein-Uhlenbeck process for the logarithm of the commodity spot price, with a reversion to a time dependent long-run level, the time variation of the long-run price level being characterized by a Fourier series. Finally, in section 5 we present some concluding remarks.

A TERM STRUCTURE MODEL WITH CYCLICAL MEAN REVERSION

In this section we introduce a continuous-time model for the term structure of interest rates assuming that the spot rate converges to a long-term level that changes over time according to a Fourier series. Under this framework, we present the partial differential equation that must be satisfied by the price of any derivative asset, obtain the bond pricing equations, and characterize the term structure of interest rates.

First, let r_t denote the instantaneous interest rate available at time t. We assume that the time evolution of r_t is given by the Ornstein-Uhlenbeck process, defined by a stochastic differential equation

$$dr_t = \kappa(f(t) - r_t)dt + \sigma dW_t \tag{1}$$

where κ, $\sigma \in R^+$ and W_t is a standard Wiener process. In addition, we assume that the mean-reversion level, $f(t)$, follows a time-dependent process driven by a Fourier series:

$$f(t) = \sum_{n=0}^{\infty} Re\left[A_n e^{in\omega t}\right]$$

where we only consider the real part of the Fourier series since it is the only one that makes economic sense. Note that, $\forall_n \mid A_n \in C$, so that there is a phase factor contained in A_n. In more detail, $A_n = A_{n,x} + iA_{n,y}$ where $A_{n,x}$, $A_{n,y} \in R$. Hence, $A_{n,x}$ and $A_{n,y}$ denote the amplitude and phase of the fluctuations in the instantaneous rate, respectively. Moreover, this model nests the model in Vasicek (1977) by taking $A_n = 0$, $\forall n \in N - \{0\}$.

Now, let $\Lambda(r_t, t)$ denote the market price of risk, which is assumed constant, $\Lambda(r_t, t) = \lambda$. Then, the risk-neutral version of the process (1) is given by

$$dr_t = \mu_r dt + \sigma d\widetilde{W}_t \tag{2}$$

where

$$\mu_r = \kappa\left(\alpha + g(t) - r_t\right) \tag{3}$$

$$\alpha = A_0 - \frac{\lambda\sigma}{\kappa} \tag{4}$$

$$g(t) = \sum_{n=1}^{\infty} Re\left[A_n e^{in\omega t}\right] = f(t) - A_0 \tag{5}$$

where $A_0 \in$ R and $\widetilde{W}_t = W_t + \lambda t$ is a standard Wiener process under the risk-neutral measure $\widetilde{P}$. The following Proposition establishes the solution of the stochastic differential equation (2).

Proposition 1. *The solution of the risk-neutral process followed by the instantaneous interest rate is given as*[1]

$$r_s = e^{-\kappa(s-t)} r_t + \left(1 - e^{-\kappa(s-t)}\right)\alpha + \sum_{n=1}^{\infty} Re\left[\frac{\kappa A_n}{\kappa + in\omega}\left(e^{in\omega s} - e^{-\kappa(s-t)+in\omega t}\right)\right]$$
$$+\sigma \int_t^s e^{-\kappa(s-u)} d\widetilde{W}_u$$

From Proposition 1, it is clear that instantaneous interest rate follows a Normal distribution. Its first two statistical moments under $\widetilde{P}$ are given as

$$\begin{aligned}\widetilde{E}[r_T \mid r_t] &= e^{-\kappa(T-t)} r_t + \left(1 - e^{-\kappa(T-t)}\right)\alpha \\ &+ \sum_{n=1}^{\infty} Re\left[\frac{\kappa A_n}{\kappa + in\omega}\left(e^{in\omega T} - e^{-\kappa(T-t)+in\omega t}\right)\right]\end{aligned} \tag{6}$$

1. This result arises as $e^{-k(s-t)}$ is square-integrable in $[t, s]$, so that it belongs to a Hilbert space.

$$
\begin{aligned}
\widetilde{V}[r_T \mid r_t] &= \widetilde{V}\left[\sigma \int_t^T e^{-\kappa(T-u)} d\widetilde{W}_u\right] = \left(\sigma \int_t^T e^{-\kappa(T-u)} d\widetilde{W}_u\right)^2 \\
&= \sigma^2 \int_t^T e^{-2\kappa(T-u)} du = \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa(T-t)}\right) \qquad (7)
\end{aligned}
$$

where we have applied the isometry property for stochastic integrals in the variance.

2.1 Bond Pricing and the Term Structure of Interest Rates

Let $P(r_t, t, T)$ denote the price at time t of a zero-coupon bond that pays \$1 at maturity T. Applying Itô's Lemma, standard no-arbitrage arguments and some trivial algebra, we get the following partial differential equation (PDE):

$$
\begin{aligned}
&P_t(r_t,t,T) + (\mu_r - \Lambda(r_t,t)\sigma_r)P_r(r_t,t,T) + \frac{1}{2}\sigma_r^2 P_{rr}(r_t,t,T) \\
&- r_t P(r_t,t,T) = 0 \qquad (8)
\end{aligned}
$$

that must be verified by the price of any derivative.

Replacing expression (1) and the constant market price of risk λ into (8), we get the PDE for the bond price:

$$
P_t + P_r \kappa\left(\alpha + g(t) - r_t\right) + P_{rr}\frac{\sigma^2}{2} - P r_t = 0 \qquad (9)
$$

subject to the terminal condition $P(r_T, T, T) = 1, \ \forall\, r_T$.

Using probabilistic techniques, the solution of this PDE can be written as a risk-neutral conditional expectation, that is,

$$
P(r_t,t,T) = \widetilde{E}\left[e^{-\int_t^T r_s ds} \mid r_t\right]
$$

Looking at Proposition 1, it is clear that $\int_t^T r_s ds$ is a random Normal variable. Then, straightforward algebra leads to the solution of this PDE as given in the following Proposition.

Proposition 2. *The price at time* t *of a zero-coupon bond with maturity T and \$1 face value is given by*

$$P(r_t, t, T) = \exp\left\{ -\widetilde{E}\left[\int_t^T r_s ds \mid r_t\right] + \frac{1}{2}\widetilde{V}\left[\int_t^T r_s ds \mid r_t\right]\right\}$$

where

$$\begin{aligned} \widetilde{E}\left[\int_t^T r_s ds \mid r_t\right] &= \frac{1-e^{-\kappa(T-t)}}{\kappa} r_t - \left(\frac{1-e^{-\kappa(T-t)}}{\kappa} - (T-t)\right)\alpha \\ &+ \sum_{n=1}^{\infty} Re\left[\frac{A_n}{n\omega(\kappa+in\omega)}\left(e^{in\omega t}\left(n\omega e^{-\kappa(T-t)} + i\kappa - n\omega\right) - i\kappa e^{in\omega T}\right)\right] \end{aligned} \quad (10)$$

$$\widetilde{V}\left[\int_t^T r_s ds \mid r_t\right] = \frac{\sigma^2}{\kappa^2}\left[(T-t) - 2\frac{1-e^{-\kappa(T-t)}}{\kappa} + \frac{1-e^{-2\kappa(T-t)}}{2\kappa}\right] \quad (11)$$

Since all affine models provide an exponential-affine functional form for bond pricing, we can immediately rewrite the previous Proposition to obtain the next one.

Proposition 3. *The price at time t of a zero-coupon bond with maturity T and \$1 face value is given by*

$$P(r_t, t, T) = e^{A(t,T) - B(t,T) r_t}$$

where

$$\begin{aligned} A(t,T) &= \frac{\sigma^2}{2\kappa^2}\left[(T-t) - 2B(t,T) + \frac{1-e^{-2\kappa(T-t)}}{2\kappa}\right] + \left(B(t,T) - (T-t)\right)\alpha \\ &- \sum_{n=1}^{\infty} Re\left[\frac{A_n}{n\omega(\kappa+in\omega)}\left(e^{in\omega t}\left(n\omega e^{-\kappa(T-t)} + i\kappa - n\omega\right) - i\kappa e^{in\omega T}\right)\right] \end{aligned} \quad (12)$$

$$B(t,T) \quad = \quad \frac{1-e^{-\kappa(T-t)}}{\kappa} \tag{13}$$

In Figure 1 we plot the term structure of bond prices for three different set of parameters in the Fourier model against the structure obtained with Vasicek's model. We can see the higher flexibility of our proposed model approach to fit different shapes of the term structure.

Under this framework and considering the bond price $P(r_t, t, T)$ given by Proposition 3, the term structure of interest rates is fully characterized in the following Corollary.

Corollary 1. *The yield to maturity, $R(r_t, t, T)$, is given by*

$$R(r_t,t,T) = -\frac{1}{\tau}\ln P(r_t,t,T), \ \tau = T-t$$

The short-term interest rate is defined as the instantaneous interest rate at time t, that is,

$$r_t = \lim_{\tau\to 0} R(r_t,t,T) = R(r_t,t,t)$$

The instantaneous forward rate is given as

$$f(r_t,t,T) = -\frac{\partial \ln(P(r_t,t,T))}{\partial T}$$

Figure 2 shows the yield curve for three different set of parameters in the Fourier model against Vasicek's model. Clearly, even for small number of terms (n) in the expansion, the Fourier model is capable of replicating different yield curve shapes such as upward sloping, downward sloping, humped, and inverted humped. On this respect, it is interesting to stress that our model should be able to replicate any yield curve shape as n goes to infinity, since the yield curve function belongs to a Hilbert space $L^2([t, T])$, and the Fourier series can be made to converge in quadratic mean to any function in such a space.

For illustrative purposes, Figures 3 and 4 show how the term structure of interest rates responds to different values of speed of reversion to the mean and the volatility parameter, respectively. Both models provide a similar pattern for the chosen parameters: the lower the speed of mean reversion, the lower the yield. Additionally, in the Fourier model, the lower the speed of mean reversion, the flatter the term structure. Moreover, Figure 4 shows that the yield decreases with volatility. Figure 5 compare how the term structure of interest rates responds in the Vasicek model and the Fourier model to different values of the common α parameter. Finally, Figure 6 displays how the term structure under the Fourier model responds to changes in its parameters $A_{n,x}$, $A_{n,y}$ and ω. The most obvious effect is that of changes in the phase $A_{n,y}$. We can see how the position and height of the peak in the term structure occur in opposite places for different phases. All these representations confirm that our proposed model provides a great flexibility even for small number of terms in the Fourier expansion.

DERIVATIVES PRICING UNDER A NEW MACRO-FINANCIAL SQUARE-ROOT PROCESS FOR THE TERM STRUCTURE OF INTEREST RATES

Unlike any other one-factor model that allow the spot rate process for time-dependent parameters (see, for instance, Hull and White (1990, 1993)), we now assume that the mean reversion level follows a cyclical process. In addition, we also consider that the interest rate volatility depends on the interest rate level. Thus, we model the behaviour of both variables assuming an harmonic oscillator as follows

$$f(t) = A\sin(\varphi - \omega t)$$

where A denotes the amplitude of the wave, ϕ the offset phase, and w the temporal frequency.

We now define the mean reversion level, θ_t, and the volatility, σ^2, as

$$\theta_t = A_\theta \sin^2(\varphi - \omega t) \tag{14}$$

$$\sigma_t^2 = A_\sigma \sin^2(\varphi - \omega t) \tag{15}$$

Hence, the positiveness of the mean reversion level and the interest rate volatility is guaranteed. Let r_t denote the instantaneous interest rate available at time t whose dynamic is

$$dr_t = \mu_r dt + \sigma_r dW_t \tag{16}$$

where W_t is a standard Wiener process and

$$\mu_r = \kappa(\theta_t - r_t) \tag{17}$$

$$\sigma_r = \sigma_t\sqrt{r_t} \tag{18}$$

where $\kappa \in R^+$. Looking at these expressions, it is clear that our model nests that presented in Cox *et al.* (1985) taking $\omega = 0$ in equations (14)-(15).

For square-root processes of this type, Cox *et al.* (1985) shows that the distribution function of interest rates is a rescaled non-central chi-square with δ degrees of freedom. Note that, whenever δ is not a positive integer, the distribution of r_t is unknown. Besides, the dimension of the process r_t is given by $\delta = \frac{4\theta_t \kappa}{\sigma_t^2}$. As both waves are in phase, the model's dimension can be represented as $\delta = \frac{4A_\theta \kappa}{A_\sigma} > 0^2$. Our model guarantees the positiveness of interest rates. On this respect, Feller (1951) studied the Fokker-Plank-Kolmogorov equation for the transition density and showed that $r_t > 0$ if $\delta \geq 2$, however it can become null if $\delta < 2$ but will never become negative.

3.1 Bond Pricing and the Term Structure of Interest Rates

This section presents closed-form expressions for the price of zero-coupon bonds and, later, we analytically compute closed-form formulas for the prices of different securities.

Let $P(r_t, t, T)$ denote the price at time t of a zero-coupon bond that pays \$1 at maturity T. Then, the bond price dynamics is given by the process

$$dP = \mu_P(r_t, t, T)P(r_t, t, T)dt + \sigma_P(r_t, t, T)P(r_t, t, T)dW_t \tag{19}$$

Applying Itô's Lemma and using (16), it can be shown that

$$\mu_P = \frac{1}{P}\left(P_t + \mu_r P_r + \frac{1}{2}\sigma_r^2 P_{rr}\right) \tag{20}$$

$$\mu_P = \frac{1}{P}\left(P_t + \mu_r P_r + \frac{1}{2}\sigma_r^2 P_{rr}\right) \tag{21}$$

2. Note that, if sin $(\phi - \omega t)$ is equal to zero, then δ becomes indeterminate. As this case would only occur for a infinitesimal period of time, we do not consider this possibility.

where arguments have been omitted and subscripts in P indicate the corresponding partial derivative. Applying standard no-arbitrage arguments, there exists a factor $\Lambda(r_t, t)$, called market price of risk, such that

$$\mu_P(r_t, t, T) - r_t = \Lambda(r_t, t)\sigma_P(r_t, t, T) \tag{22}$$

Finally, some trivial algebra leads to the following partial differential equation (PDE)

$$\begin{aligned} &P_t(r_t, t, T) + (\mu_r - \Lambda(r_t, t)\sigma_r)P_r(r_t, t, T) + \frac{1}{2}\sigma_r^2 P_{rr}(r_t, t, T) \\ &- r_t P(r_t, t, T) = 0 \end{aligned} \tag{23}$$

that must be verified by the price of any derivative.

Considering a market price of risk such as

$$\Lambda(r_t, t) = \frac{\lambda_t \sqrt{r_t}}{\sigma_t} \tag{24}$$

Using expressions (18)-(24), the PDE (23) becomes

$$\begin{aligned} &P_t(r_t, t, T) + (\kappa(\theta_t - r_t) - \lambda_t r_t)P_r(r_t, t, T) + \frac{1}{2}\sigma_t^2 r_t P_{rr}(r_t, t, T) \\ &- r_t P(r_t, t, T) = 0 \end{aligned} \tag{25}$$

The solution of this PDE, subject to the boundary condition $P(r_T, T, T) = 1$, $\forall r_T$, is given by the following Proposition.

Proposition 4. *The price at time* t *of a zero-coupon bond with maturity T and \$1 face value is given by*

$$P(r_t, \tau) = A(\tau)e^{-B(\tau)r_t}$$

where

$$
\begin{aligned}
A(\tau) &= \exp\left\{-\int_t^T \kappa\theta_t B(\tau)dt\right\} \\
B(\tau) &= \frac{c_1 MC(a,q,x) + MS(a,q,x)}{\frac{1}{2}(\lambda+\kappa)\left(c_1 MC(a,q,x) + MS(a,q,x)\right) + \omega\left(c_1 MCP(a,q,x) + MSP(a,q,x)\right)}
\end{aligned}
$$

$$
\begin{aligned}
a &= -\frac{A_\sigma + (\lambda+\kappa)^2}{4\omega^2} \\
q &= -\frac{A_\sigma}{8\omega^2} \\
x &= \varphi - \omega t \\
c_1 &= -\frac{MS(a,q,\varphi-\omega T)}{MC(a,q,\varphi-\omega T)} \\
\tau &= T - t
\end{aligned}
$$

where θ_t is given by (14), MC and MS represent the Mathieu cosine and sine function, respectively, and MCP and MSP represent the derivative with respect to x of the Mathieu cosine and sine function, respectively.

Figure 7 compares the bond price in the CIR model against three alternatives in our model. We check that, in our model, the bond price does not decrease monotonically with time to matu- rity. Additionally, we provide much more flexibility than the CIR model with the same analytical tractability. We can also visualize the presence of humps, which is a very desirable effect not only here but also in any interest rate derivative.

Corollary 2. *The yield to maturity, $R(r_t$, t, $T)$, is given by*

$$
R(r_t, t, T) = -\frac{1}{\tau}\ln P(r_t, t, T), \;\; \tau = T - t
$$

The short-term interest rate is defined as the instantaneous interest rate at time t, that is,

$$
r_t = \lim_{\tau\to 0} R(r_t, t, T) = R(r_t, t, t)
$$

The instantaneous forward rate is given as

$$f(r_t, t, T) = -\frac{\partial \ln(P(r_t, t, T))}{\partial T}$$

Figure 8 shows the term structure of interest rates in the CIR model and three alternatives in our model. We can see how our model adds flexibility as we can reflect different behaviours for the term structure.

For illustrative purposes, Figures 9 and 10 show how the term structure of interest rates responds to changes in the mean reversion speed and volatility in both models. In the CIR model, the higher the speed of mean reversion, the higher the interest rate while, in our model, the lower the mean reversion speed, the flatter the term structure. Besides, in our model, there is a twist in the pattern due to the cyclic behaviour. In Figure 10, for both models, the higher the volatility, the lower the term structure.

Figures 11 and 12 reflect the response of the term structure of interest rates to different values of the mean reversion level in both models. In the CIR model, the higher the mean reversion level, the higher the yield. In our model, it is harder to analyse this response as it depends on three parameters. Anyway, we observe that the lower the amplitude, the flatter and the lower the term structure. When changing the temporal frequency, it seems clear that the higher the temporal frequency, the more humped the term structure. Finally, for different offset phases, the curves occasionally crossover each other.

VALUATION OF COMMODITY DERIVATIVES WHEN SPOT PRICES REVERT TO A CYCLICAL MEAN

In this section we analyse the commodity market. In more detail, given the seasonal behaviour exhibited by most commodities, this section introduces a continuous-time model based on an Ornstein-Uhlenbeck process for the logarithm of the commodity spot price, with a reversion to a time dependent long-run level, the time variation of the long-run price level being characterized by a Fourier series.

Let S_t denote the commodity spot price available at time t. Then, the evolution of the commodity spot price, S_t, is given by the stochastic differential equation

$$dS_t = \kappa \left(f(t) - \ln(S_t) \right) S_t dt + \sigma S_t dW_t \tag{26}$$

where κ, $\sigma \in \mathrm{R}^+$ and W_t is a standard Wiener process. The main assumption made in this model is that the mean reversion level, $f(t)$, follows a time-dependent periodic function characterized by a Fourier series, in more detail

$$f(t) = \sum_{n=0}^{\infty} Re\left[A_n e^{inwt}\right]$$

where it is only considered the real part of the series since it is the part that makes economic sense. Note that, $\forall n \mid A_n \in \mathrm{C}$, so that there is a phase factor contained in A_n. In more detail, consider $A_n = A_{x,n} + iA_{y,n}$ where $A_{x,n}$, $A_{y,n} \in \mathrm{R}$. Hence, $A_{x,n}$ and $A_{y,n}$ denote the amplitude and phase of each term in the Fourier expansion, respectively. Note that this model nests model 1 presented in Schwartz (1997) by taking $A_n = 0$, $\forall n \in \mathrm{N} - \{0\}$.

Moreover, defining $X_t = \ln(S_t)$, assuming a constant market price of risk, that is $\Lambda(S_t, t) = \lambda$, and applying Ito's Lemma, the log price can be represented by the following risk-neutral process

$$dX_t = \mu_t dt + \sigma d\widetilde{W}_t \tag{27}$$

where

$$\mu_t = \kappa\left(\widetilde{\alpha} + g(t) - X_t\right) \tag{28}$$

$$\widetilde{\alpha} = A_0 - \frac{\sigma^2}{2\kappa} - \frac{\lambda\sigma}{\kappa} \tag{29}$$

$$g(t) = \sum_{n=1}^{\infty} Re\left[A_n e^{inwt}\right] \tag{30}$$

where $A_0 \in$ R and $\widetilde{W}_t = W_t + \lambda t$ is a standard Wiener process under the risk-neutral measure $\widetilde{P}$.

The following Proposition establishes the solution of the stochastic differential equation (27).

Proposition 5. *The solution of the risk-neutral process followed by the logarithm of the commodity spot price is given as*

$$X_s = e^{-\kappa(s-t)} X_t + \left(1 - e^{-\kappa(s-t)}\right)\widetilde{\alpha} + \sum_{n=1}^{\infty} Re\left[\frac{\kappa A_n}{\kappa + inw}\left(e^{inws} - e^{-\kappa(s-t)+inwt}\right)\right]$$
$$+\sigma \int_t^s e^{-\kappa(s-u)} d\widetilde{W}_u$$

Figure 13 presents the evolution of the spot price time series for four different set of parameters. In the first graph we only consider the drift process, that is $\sigma = 0$. We can see how flexible this model is, in fact, any scenario can be replicated increasing the number of terms in the Fourier expansion. The second graph considers the drift and diffusion process, this representation presents a simulated spot price walk considering each underlying scenario. For illustrative purposes, Figures 14 and 15 show how the spot price responds to different values of $\widetilde{\alpha}$, κ, $A_{n,x}$, $A_{n,y}$, and ω with $n = 1$, $\sigma = 0$.

From Proposition 5, it is clear that the conditional distribution of the logarithm of the commodity spot price at time T follows a normal distribution where the mean and variance under the risk-neutral probability measure $\widetilde{P}$ are given as

$$\widetilde{E}\left[X_T|F_t\right] = e^{-\kappa(s-t)}X_t + \left(1-e^{-\kappa(s-t)}\right)\widetilde{\alpha} + \sum_{n=1}^{\infty} Re\left[\frac{\kappa A_n}{\kappa+inw}\left(e^{inws}-e^{-\kappa(s-t)+inwt}\right)\right] \quad (31)$$

$$\widetilde{V}\left[X_T|F_t\right] = \widetilde{V}\left[\sigma\int_t^T e^{-\kappa(T-u)}d\widetilde{W}_u\right] = \left(\sigma\int_t^T e^{-\kappa(T-u)}d\widetilde{W}_u\right)^2 = \sigma^2\int_t^T e^{-2\kappa(T-u)}du = \frac{\sigma^2}{2\kappa}\left(1-e^{-2\kappa(T-t)}\right) \quad (32)$$

where we have applied the isometry property for stochastic integrals in the variance.

Since $X_t = \ln(S_t)$, the forward price of a commodity maturing at time T is a straightforward application of the properties of the log-normal distribution under the risk-neutral measure. Hence, the following proposition arises.

Proposition 6. *Assuming a constant interest rate, the forward price of a commodity maturing at time T is given by*

$$\begin{aligned} F(S_t,t,T) &= \widetilde{E}\left[S_T|F_t\right] = \exp\left\{\widetilde{E}\left[X_T|F_t\right] + \frac{1}{2}\widetilde{V}\left[X_T|F_t\right]\right\} \\ &= \exp\left\{e^{-\kappa(T-t)}\ln(S_t) + \left(1-e^{-\kappa(T-t)}\right)\widetilde{\alpha} + \frac{\sigma^2}{4\kappa}\left(1-e^{-2\kappa(T-t)}\right)\right. \\ &\quad \left. + \sum_{n=1}^{\infty} Re\left[\frac{\kappa A_n}{\kappa+inw}\left(e^{inws}-e^{-\kappa(s-t)+inwt}\right)\right]\right\} \end{aligned}$$

Alternatively,

$$\ln(F(S_t, t, T)) = e^{-\kappa(T-t)} \ln(S_t) + \left(1 - e^{-\kappa(T-t)}\right) \widetilde{\alpha} + \frac{\sigma^2}{4\kappa} \left(1 - e^{-2\kappa(T-t)}\right) + \sum_{n=1}^{\infty} Re\left[\frac{\kappa A_n}{\kappa + inw} \left(e^{inws} - e^{-\kappa(s-t)+inwt}\right)\right] \quad (33)$$

4.1 Option Pricing

This section focuses on option pricing. In more detail, we compute closed-form expressions for the prices of European options written on the commodity and the forward commodity price under the new model framework.

European option on the commodity

Consider a call option maturing at time T with strike K, written on a commodity. Let $c_t(S_t;t;T;K)$ denote the price at time t of this call option. Then, the terminal condition to this call option is given by

$$c_T(S_T;T;T;K) = \max\{F(S_T;T;T) - K;0\}$$

Hence, under the risk-neutral measure $\widetilde{P}$, the price at time t of this option will be given by

$$c_t(S_t;t;T;K) = \widetilde{E}\left[e^{-r(T-t)}(F(S_t;t;T) - K)^{+}|F_t\right]$$

The call option price is given by the following Proposition.

Proposition 7. *The price at time t of a European call option with maturity T written on a commodity is given by*

$$\begin{aligned} c_t(S_t, t, T, K) &= \widetilde{E}\left[e^{-r(T-t)}(S_T - K)^{+}|F_t\right] \\ &= e^{-r(T-t)} \int_{-\infty}^{\infty} (S_T - K)^{+} \rho(\mu, \Sigma) dX_T \\ &= e^{-r(T-t)} \left[e^{\mu + \frac{1}{2}\Sigma^2} \Phi(d_1) - K\Phi(d_2)\right] \end{aligned}$$

where $\rho(\mu, \Sigma)$ *defines the normal density function and*

$$\begin{aligned}
\mu &= \widetilde{E}[X_T|F_t] \\
\Sigma &= \widetilde{V}[X_T|F_t] \\
d_1 &= \frac{\mu + \Sigma^2 - \ln(K)}{\Sigma} \\
d_2 &= d_1 - \Sigma
\end{aligned}$$

with $\tilde{E}[X_T|F_t]$ *and* $\tilde{V}[X_T|F_t]$ *given by equation (31) and (32), respectively.*

European option on the commodity forward

Consider a European forward call option that matures at time T with strike K. If this option is exercised, the call-holder pays K and receives a forward maturing at time s on a commodity. Let $c_t(S_t;$ t$; T; s; K)$ denote the price at time t of this option. The terminal condition of this option is given as

$$c_T(S_T; T; s; K) = max\{F(S_T; T; s) - K, 0\}$$

Under the risk-neutral measure $\tilde{P}$, the price at time t of this option is given as

$$c_t(S_t; t; T; s; K) = \widetilde{E}\left[e^{-r(T-t)}(F(S_T; T; s) - K)^+|F_t\right]$$

Hence, the following proposition arises.

Proposition 8. *The price at time t of a European forward call option with maturity T on a forward contract expiring at time s written on a commodity is given by*

$$\begin{aligned}
c(S_t, t, T, s, K) = &\ \widetilde{E}\left[e^{-r(T-t)}(F(S_T, T, s) - K)^+|F_t\right] \\
= &\ e^{-r(T-t)} \int_{-\infty}^{\infty} (F(S_T, T, s) - K)^+ \rho(\mu, \Sigma) dX_T \\
= &\ e^{-r(T-t)} \left[\exp\left\{\Omega + \mu e^{-\kappa(s-T)} + \frac{1}{2}\Sigma^2 e^{-2\kappa(s-T)}\right\}\Phi(d_1) - K\Phi(d_2)\right]
\end{aligned}$$

where $\rho(\mu, \Sigma)$ denotes the normal density function and

$$
\begin{aligned}
\mu &= \widetilde{E}\left[X_T|F_t\right] \\
\Sigma^2 &= \widetilde{V}\left[X_T|F_t\right] \\
\Omega &= \left(1-e^{-\kappa(s-T)}\right)\widetilde{\alpha}+\left(1-e^{-2\kappa(s-T)}\right)\frac{\sigma^2}{4\kappa} \\
&\quad +\sum_{n=1}^{\infty} Re\left[\frac{\kappa A_n}{\kappa+inw}\left(e^{inws}-e^{-\kappa(s-T)+inwT}\right)\right] \\
\nu &= \left(\ln(K)-\Omega\right)e^{\kappa(s-T)} \\
d_1 &= \frac{\mu+\Sigma^2 e^{-\kappa(s-T)}-\nu}{\Sigma} \\
d_2 &= \frac{\mu-\nu}{\Sigma}
\end{aligned}
$$

with $\tilde{E}[X_T|F_t]$ and $\tilde{V}[X_T|F_t]$ given by equation (31) and (32), respectively.

CONCLUSIONS

Characterizing the stochastic behaviour of interest rates and commodity prices constitute an issue of special relevance for practitioners in financial markets and it has been deeply analysed in many academic papers throughout the years. In this work we have introduced three different continuous- time models allowing the underlying state variable to capture any possible seasonal or cyclical behaviour.

Firstly, in section 2, we have presented a model for the term structure of interest rates assuming that instantaneous spot rate converges to a certain time-dependent long term level that varies over time according to a Fourier series. In section 3 we dig deeper into the term structure of interest rates assuming that the spot rate follows a square-root process where both the mean reversion level and the volatility parameter change over time as a sinusoidal function. Finally, given the seasonal behaviour exhibited by most commodities, section 4 analyses the commodity market. In a similar fashion as in section 2, we assume that the logarithm of the commodity spot price follows an Ornstein-Uhlenbeck process with a reversion to a time dependent long-run level characterized by a Fourier series.

The results obtained have strong practical applications, each model fulfils a real necessity providing a powerful and simple tool for pricing and risk management purposes and should be of special interest for traders, financial institutions, and risk managers.

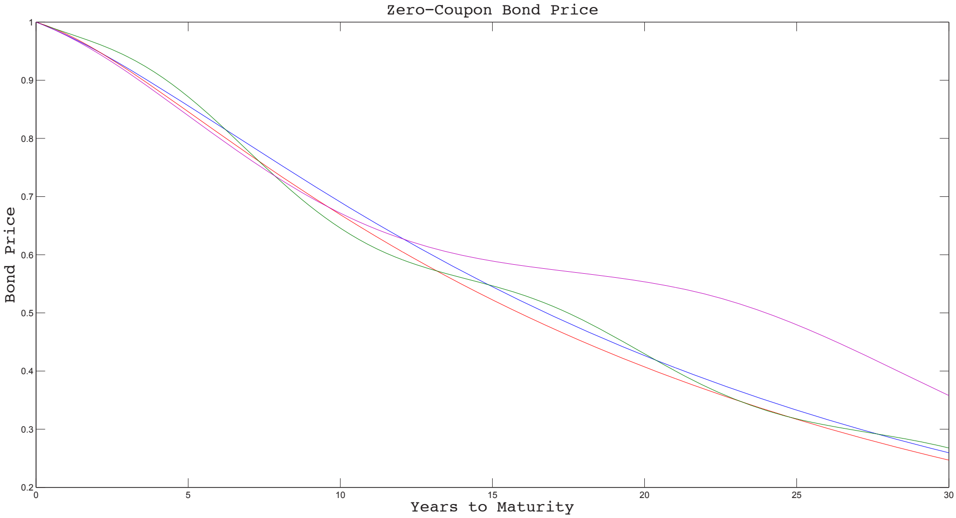
APPENDIX OF TABLES

Table I: Term Structure Models

Author(s)	Model Specification	
Merton (1973)	$dr = \theta dt + \sigma dw$	θ, σ are constant
Vasicek (1977)	$dr = \kappa(\theta - r)dt + \sigma dw$	κ, θ, σ are constant
Cox *et al.* (1985)	$dr = \kappa(\theta - r)dt + \sigma\sqrt{r}dw$	κ, θ, σ are constant
Chan *et al.* (1992)	$dr = \kappa(\theta - r)dt + \sigma r^{\gamma}dw$	κ, θ, σ, γ are constant
Ho and Lee (1986)	$dr = \theta t\ dt + \sigma dw$	θ_t is time-varying and σ is constant
Black *et al.* (1990)	$d\ \mathrm{In}(r) = \left[\theta_t - \frac{\sigma'_t}{\sigma_t}\right]dt + \sigma_t dw$	θ_t, σ_t are time-varying
Hull and White (1990, 1993)	$dr = \kappa(\theta_t - r)dt + \sigma_t r^{\gamma}dw$	θ_t, σ_t are time-varying, $\gamma = 0, 1/2$
Black and Karasinski (1991)	$d\ln(r) = \varphi_t\ [\ln(\mu_t) - \ln(r)]dt + \sigma_t dw$	φ_t, μ_t are time-varying
Heath *et al.* (1992)	$df = \alpha_t dt + \sigma_t dw$	f is the forward rate
Mercurio and Moraleda (2000)	$dr = r\left[\eta_t - \left(\lambda - \frac{\gamma}{1+\gamma t}\right)\mathrm{In}(r)\right]dt + \sigma r dw$	η_t is time-varying and λ, γ, σ are constant
Brennan and Schwartz (1979)	$dr = \theta_r dt + \sigma_{r1}\ dw_1 + \sigma_{r2}\ dw_2$ $dl = \theta_l dt + \sigma_{l1}\ dw_1 + \sigma_{l2}\ dw_2$	θ_i, σ_{ij}, $i = r, l$, $j = 1,2$ are constant
Schaefer and Schwartz (1984)	$ds = m(\mu - s)dt + \eta dw_1$ $dl = (\sigma^2 - ls)dt + \sigma\sqrt{l}dw_2$	m, μ, η, σ are constant
Longstaff and Schwartz (1992)	$dx = (\gamma - \delta x)dt + \sqrt{x}dw_1$ $dy = (\eta - vy)dt + \sqrt{x}dw_2$	γ, δ, η, v are constant
Duffie and Kan (1996)	$dX_1 = (b_1 + \Sigma_{i=1}^{2}\ a_{1i}X_i)dt + \sigma_{11}\sqrt{\alpha_1 + \Sigma_{i=1}^{2}\ \beta_{1i}X_i}\,dw_1$ $dX_2 = (b_2 + \Sigma_{i=1}^{2}\ a_{2i}X_i)dt + \sigma_{22}\sqrt{\alpha_2 + \Sigma_{i=1}^{2}\ \beta_{2i}X_i}\,dw_2$	X_i, $i = 1, 2$ are the yields of two zero-coupon bonds
Chen (1996)	$dr = \kappa(\theta - r)dt + \sqrt{\sigma}\sqrt{r}dw_1$ $d\theta = v(\hat{\theta} - \theta)dt + \varsigma\sqrt{\theta}dw_2$ $d\sigma = \mu(\hat{\sigma} - \sigma)dt + \eta\sqrt{\sigma}dw_3$	κ, v, $\hat{\theta}$, ς, μ, $\hat{\sigma}$, η are constant

APPENDIX OF FIGURES

Figure I: Simulation of the Zero-coupon bond price term structure for an arbitrary set of parameters



Parameter Values Vasicek Model:

Blue line: $r_0 = 0{,}02$; $\alpha = 0{,}05$; $\sigma = 0{,}002$; $\kappa = 0{,}2$.

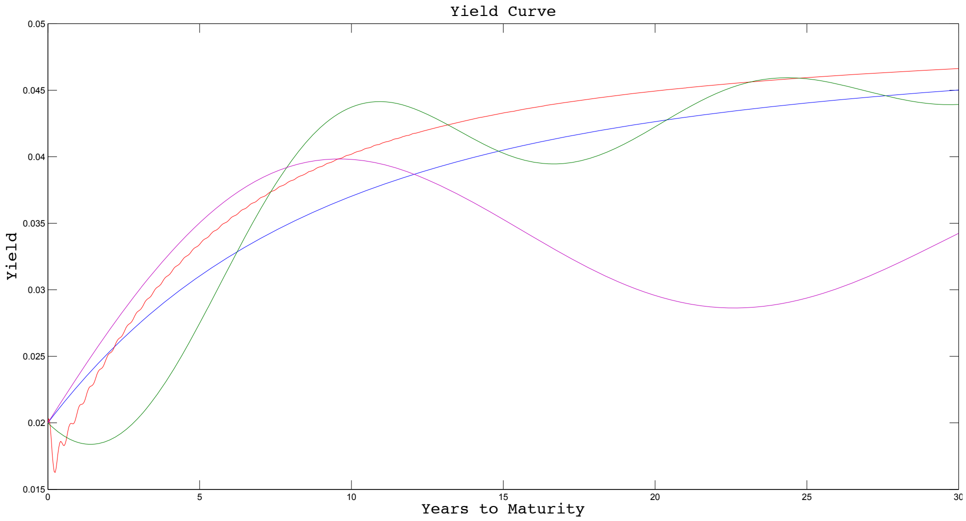
Parameter Values Fourier Model:

Red line: $r_0 = 0{,}02$; $\alpha = 0{,}05$; $\sigma = 0{,}0011$; $\kappa = 0{,}3397$; $\omega = 20$; $n = 5$; $A_{1,x} = 0{,}1758$; $A_{1,y} = 0{,}0402$; $A_{2,x} = -0{,}3011$; $A_{2,y} = 0{,}0172$; $A_{3,x} = 0{,}0498$; $A_{3,y} = -0{,}1215$; $A_{4,x} = 0{,}0798$; $A_{4,y} = 0{,}1618$; $A_{5,x} = 0{,}0894$; $A_{5,y} = 0{,}0655$.

Green line: $r_0 = 0{,}02$; $\alpha = 0{,}07$; $\sigma = 0{,}0005$; $\kappa = 0{,}018$; $\omega = 0{,}48$; $n = 2$; $A_{1,x} = -1{,}8$; $A_{1,y} = 1$; $A_{2,x} = 1{,}5$; $A_{2,y} = -1{,}5$.

Violet line: $r_0 = 0{,}02$; $\alpha = 0{,}08$; $\sigma = 0{,}0002$; $\kappa = 0{,}02$; $\omega = 0{,}25$; $n = 1$; $A_{1,x} = 0{,}3$; $A_{1,y} = 0{,}03$.

Figure 2: Term Structure of Interest Rates for an arbitrary set of parameters



Parameter Values Vasicek Model:

Blue line: $r_0 = 0{,}02$; $\alpha = 0{,}05$; $\sigma = 0{,}002$; $\kappa = 0{,}2$.

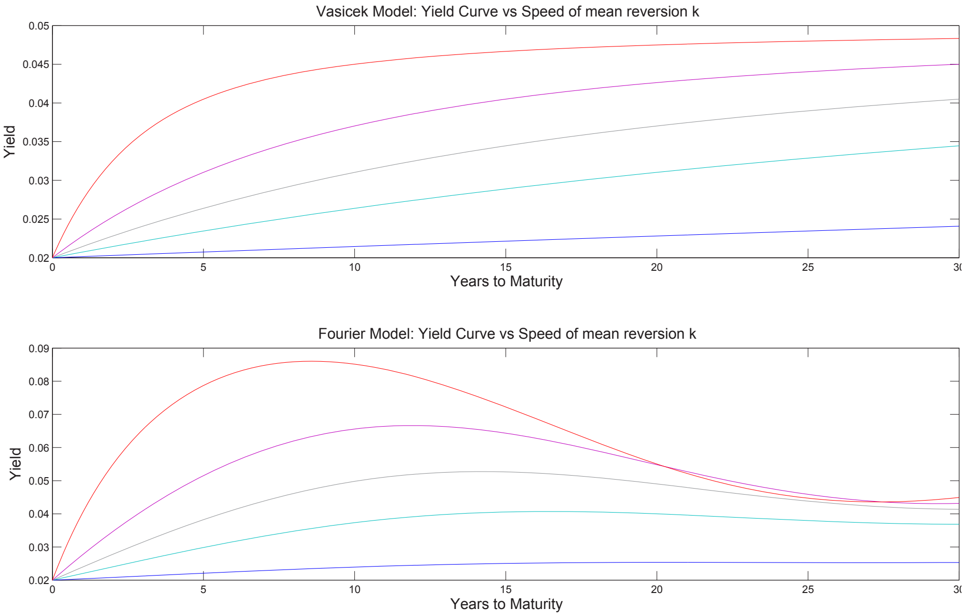
Parameter Values Fourier Model:

Red line: $r_0 = 0{,}02$; $\alpha = 0{,}05$; $\sigma = 0{,}0011$; $\kappa = 0{,}3397$; $\omega = 20$; $n = 5$; $A_{1,x} = 0{,}1758$; $A_{1,y} = 0{,}0402$; $A_{2,x} = -0{,}3011$; $A_{2,y} = 0{,}0172$; $A_{3,x} = 0{,}0498$; $A_{3,y} = -0{,}1215$; $A_{4,x} = 0{,}0798$; $A_{4,y} = 0{,}1618$; $A_{5,x} = 0{,}0894$; $A_{5,y} = 0{,}0655$.

Green line: $r_0 = 0{,}02$; $\alpha = 0{,}07$; $\sigma = 0{,}0005$; $\kappa = 0{,}018$; $\omega = 0{,}48$; $n = 2$; $A_{1,x} = -1{,}8$; $A_{1,y} = 1$; $A_{2,x} = 1{,}5$; $A_{2,y} = -1{,}5$.

Violet line: $r_0 = 0{,}02$; $\alpha = 0{,}08$; $\sigma = 0{,}0002$; $\kappa = 0{,}02$; $\omega = 0{,}25$; $n = 1$; $A_{1,x} = 0{,}3$; $A_{1,y} = 0{,}03$.

Figure 3: Term structure of interest rates for different values of the speed of mean reversion κ

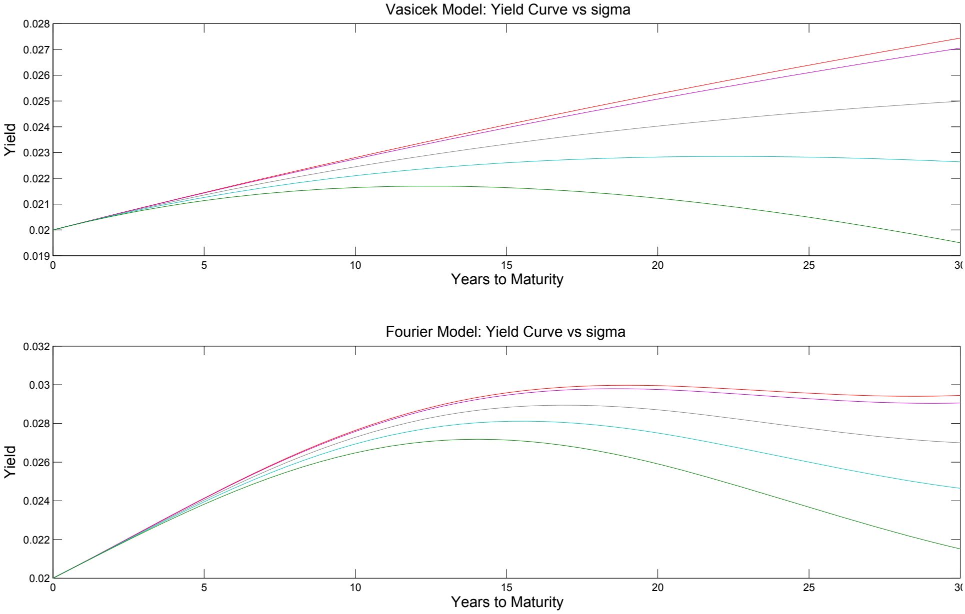


In both models, the values of κ corresponding to the curves from the top down are 0,6; 0,2; 0,1; 0,05 and 0,01 respectively

Parameter Values Vasicek Model : $r_0 = 0{,}02$; $\alpha = 0{,}05$; $\sigma = 0{,}0002$.

Parameter Values Fourier Model : $r_0 = 0{,}02$; $\alpha = 0{,}05$; $\sigma = 0{,}0002$; $\omega = 0{,}20$; $n = 1$; $A_{1,x} = 0{,}05$; $A_{1,y} = -0{,}03$.

Figure 4: Term structure of interest rates for different values of σ

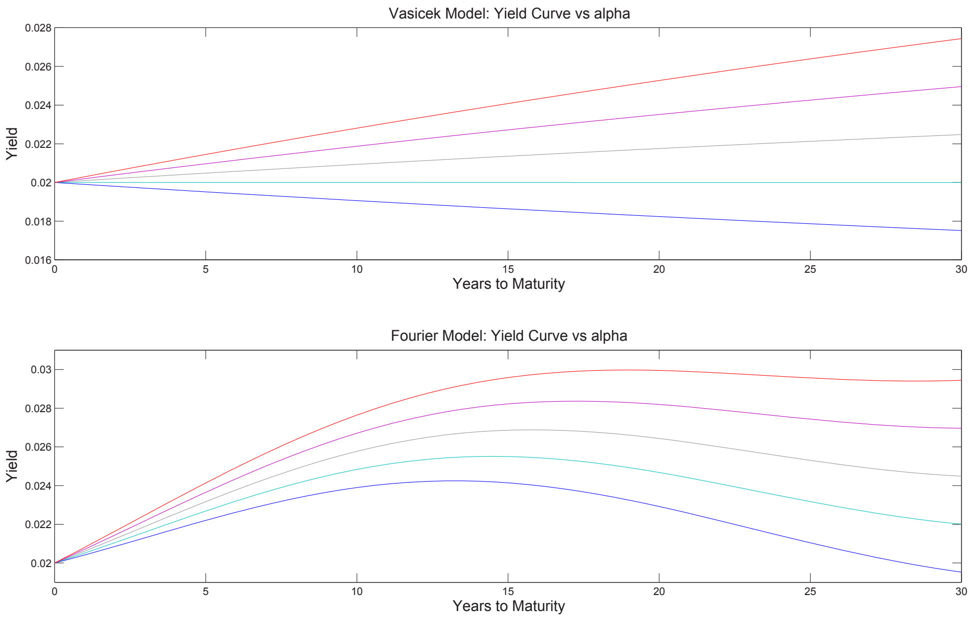


In both models, the values of σ corresponding to the curves from the top down are 0,0002; 0,002; 0,005; 0,007 and 0,009 respectively

Parameter Values Vasicek Model: $r_0 = 0{,}02$; $\alpha = 0{,}05$; $\kappa = 0{,}02$.

Parameter Values Fourier Model: $r_0 = 0{,}02$; $\alpha = 0{,}05$; $\kappa = 0{,}02$; $\omega = 0{,}20$; $n = 1$; $A_{1,x} = 0{,}05$; $A_{1,y} = -0{,}03$.

Figure 5: Term structure of interest rates for different values of the mean reversion level α

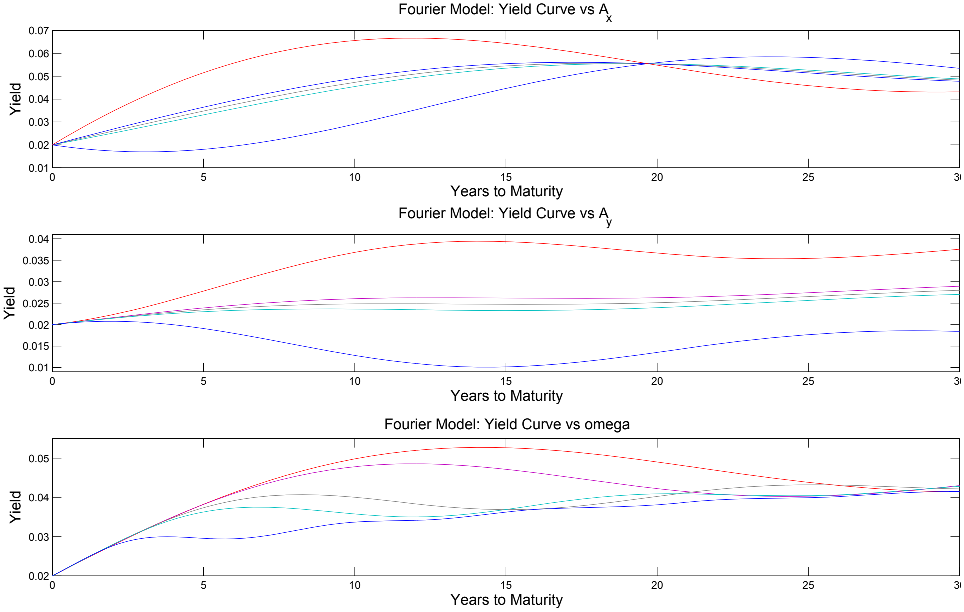


In both models, the values of α corresponding to the curves from the top down are 0,05; 0,04; 0,03; 0,02 and 0,01 respectively.

Parameter Values Vasicek Model : $r_0 = 0{,}02$; $\sigma = 0{,}0002$; $\kappa = 0{,}02$.

Parameter Values Fourier Model : $r_0 = 0{,}02$; $\sigma = 0{,}0002$; $\kappa = 0{,}02$; $\omega = 0{,}20$; $n = 1$; $A_{1,x} = 0{,}05$; $A_{1,y} = -0{,}03$.

Figure 6: Term structure of interest rates for different values of the Fourier parameters

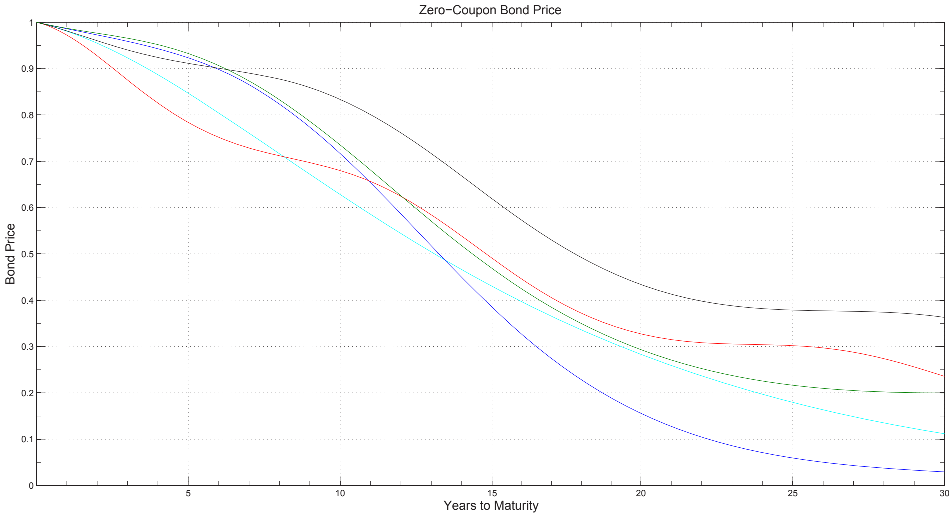


First Graph: $r_0 = 0{,}02$; $\alpha = 0{,}05$; $\sigma = 0{,}0002$; $\kappa = 0{,}2$; $\omega = 0{,}20$; $n = 1$; $A_{1,y} = -0{,}03$; and $A_{1,x} = -0{,}05; -0{,}005; 0; 0{,}005; 0{,}05$.

Second Graph: $r_0 = 0{,}02$, $\alpha = 0{,}05$, $\sigma = 0{,}0002$, $\kappa = 0{,}02$, $\omega = 0{,}30$, $n = 1$, $A_{1,x} = 0{,}05$. and $A_{1,y} = -0{,}2; -0{,}02; 0; 0{,}02; 0{,}2$.

Third Graph: $r_0 = 0{,}02$, $\alpha = 0{,}05$, $\sigma = 0{,}0002$, $\kappa = 0{,}1$, $n = 1$, $A_{1,x} = 0{,}05$, $A_{1,y} = -0{,}03$ and $\omega = 0{,}2; 0{,}25; 0{,}4; 0{,}5; 1$.

Figure 7: Simulation of the Zero-coupon bond price term structure for an arbitrary set of parameters



Parameter Values CIR Model:

Lightblue line: $r_0 = 0{,}015$; $\theta = 0{,}1$; $\sigma = 0{,}005$; $\kappa = 0{,}1$.

Parameter Values Cyclic Model:

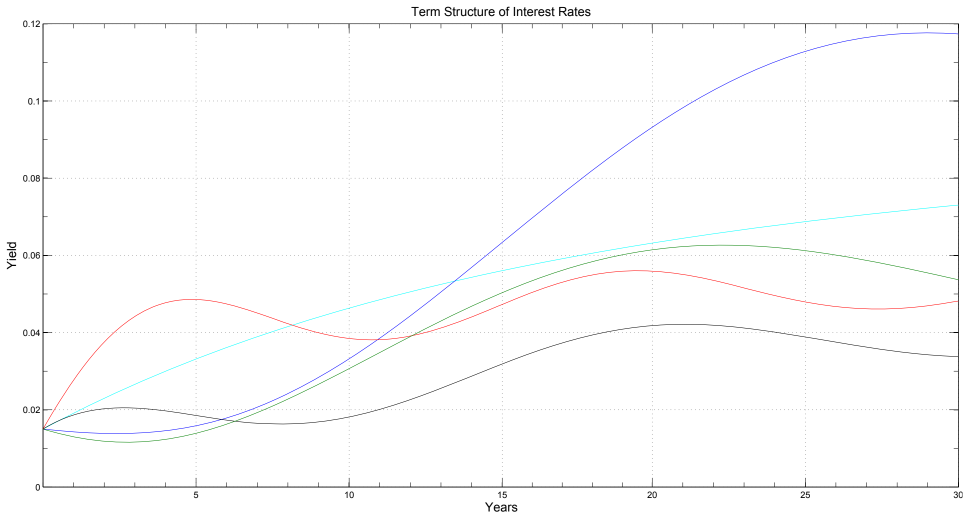
Blue line: $r_0 = 0{,}015$; $A\theta = 0{,}2$; $A\sigma = 0{,}001$; $\kappa = 0{,}1$; $\omega = 0{,}08$; $\phi = \pi$; $\lambda = 0$.

Red line: $r_0 = 0{,}015$; $A\theta = 0{,}1$; $A\sigma = 0{,}005$; $\kappa = 0{,}15$; $\omega = 0{,}2$; $\phi = \pi/2$; $\lambda = 0$.

Black line: $r_0 = 0{,}015$; $A\theta = 0{,}08$; $A\sigma = 0{,}002$; $\kappa = 0{,}15$; $\omega = 0{,}15$; $\phi = \pi/4$; $\lambda = 0$.

Green line: $r_0 = 0{,}015$; $A\theta = 0{,}1$; $A\sigma = 0{,}002$; $\kappa = 0{,}3$; $\omega = 0{,}10$; $\phi = \varpi$; $\lambda = 0$.

Figure 8: Term Structure of Interest Rates for an arbitrary set of parameters



Parameter Values CIR Model:

Lightblue line: $r_0 = 0{,}015$; $\theta = 0{,}1$; $\sigma = 0{,}005$; $\kappa = 0{,}1$

Parameter Values Cyclic Model:

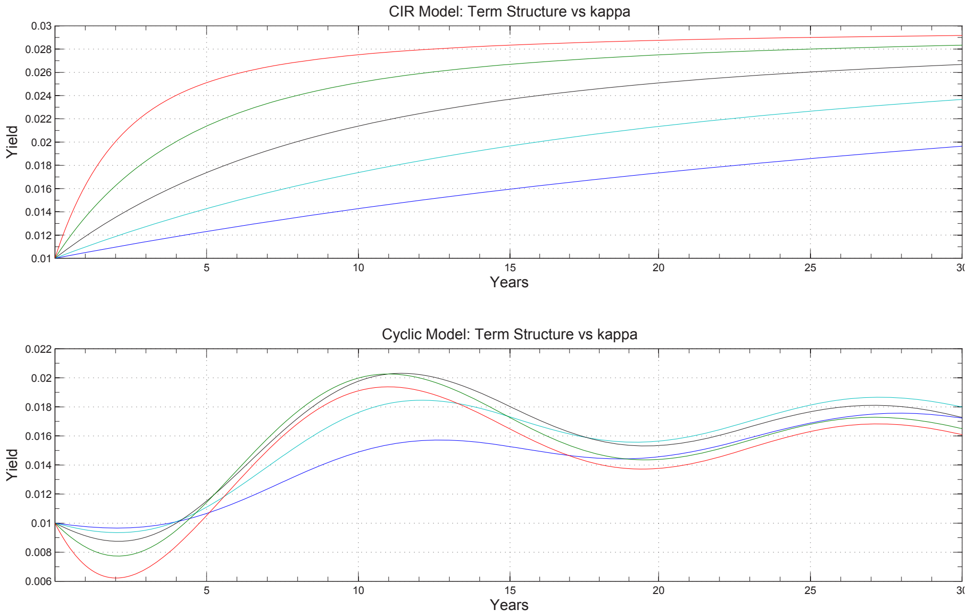
Blue line: $r_0 = 0{,}015$; $A\theta = 0{,}2$; $A\sigma = 0{,}001$; $\kappa = 0{,}1$; $\omega = 0{,}08$; $\phi = \pi$; $\lambda = 0$.

Red line: $r_0 = 0{,}015$; $A\theta = 0{,}1$; $A\sigma = 0{,}005$; $\kappa = 0{,}15$; $\omega = 0{,}2$; $\phi = \pi/2$; $\lambda = 0$.

Black line: $r_0 = 0{,}015$; $A\theta = 0{,}08$; $A\sigma = 0{,}002$; $\kappa = 0{,}15$; $\omega = 0{,}15$; $\phi = \pi/4$; $\lambda = 0$.

Green line: $r_0 = 0{,}015$; $A\theta = 0{,}1$; $A\sigma = 0{,}002$; $\kappa = 0{,}3$; $\omega = 0{,}10$; $\phi = \varpi$; $\lambda = 0$.

Figure 9: Term structure of interest rates for different values of the speed of mean reversion κ



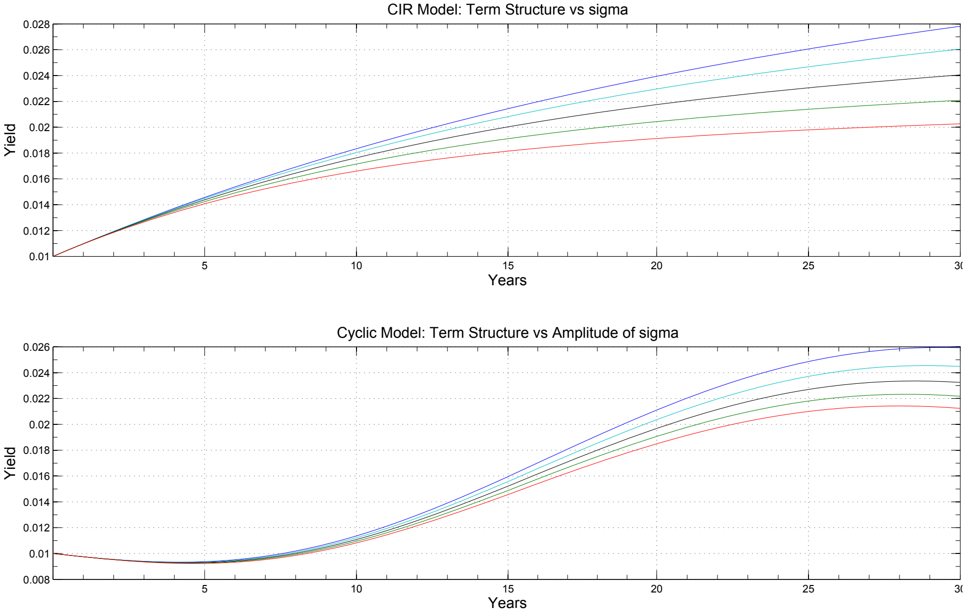
In both models, the values of κ are:

Blue Line: = 0,05; *Lightblue Line*: = 0,1; *Black Line*: = 0,2; *Green Line*: = 0,4; and *Red Line*: = 0,8.

Parameter Values CIR Model: r_0 = 0,01; θ= 0,03; σ = 0,0002.

Parameter Values Cyclic Model: r_0 = 0,01; A_θ= 0,03; A_σ = 0,0002; ω = 0,20; ϕ = 0; λ = 0.

Figure 10: Term structure of interest rates for different values of the volatility parameter

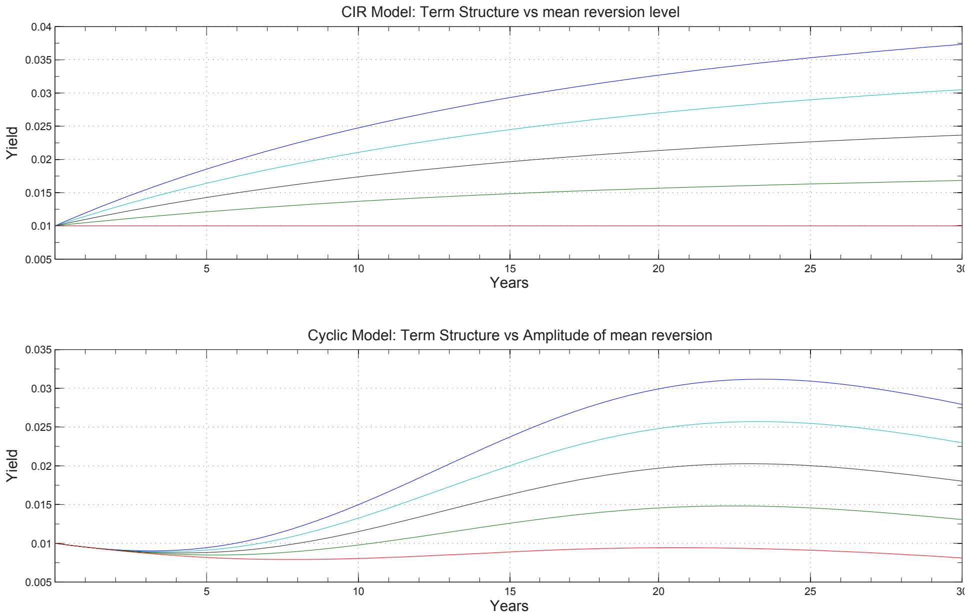


The values of σ and A_σ corresponding to CIR and the Cyclic model, respectively, are: *Blue Line*: = 0,003; *Lightblue Line*: = 0,005; *Black Line*: = 0,007; *Green Line*: = 0,009; and *Red Line*:= 0,011.

Parameter Values CIR Model: r_0 = 0,01; θ= 0,05; κ = 0,05.

Parameter Values Cyclic Model: r_0 = 0,01; A_θ= 0,05; κ = 0,05; ω = 0,08; ϕ = 0; λ = 0.

Figure II: Term structure of interest rates for different values of the mean reversion level

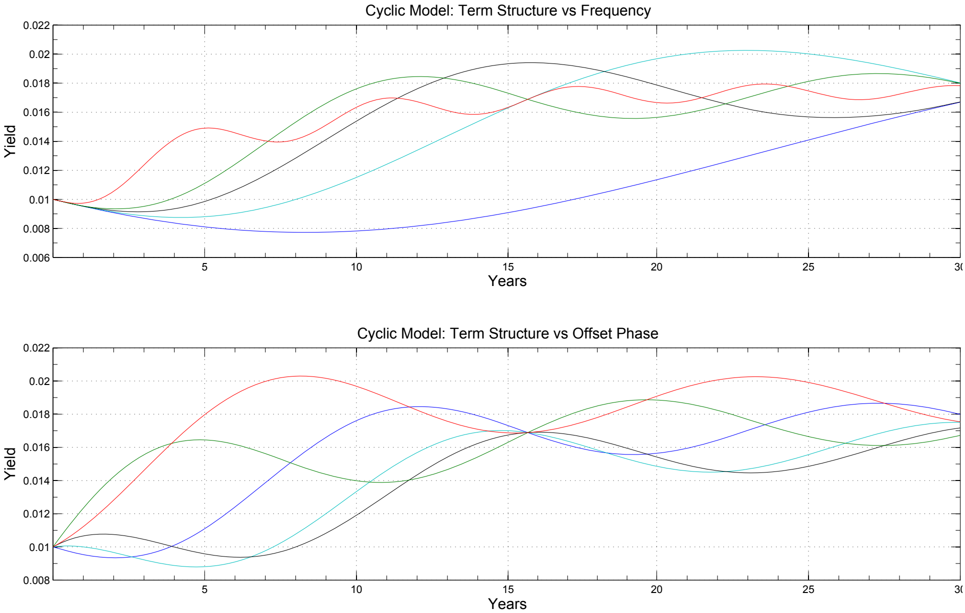


The values of θ and A_θ corresponding to CIR and the Cyclic model, respectively, are: *Blue Line*: = 0,05; *Lightblue Line*: = 0,04; *Black Line*: = 0,03; *Green Line*: = 0,02; and *Red Line*:= 0,01.

Parameter Values CIR Model: $r_0 = 0{,}01$; $\sigma = 0{,}0002$; $\kappa = 0{,}1$,

Parameter Values Cyclic Model: $r_0 = 0{,}01$; $A_\sigma = 0{,}0002$; $\kappa = 0{,}1$; $\omega = 0{,}1$; $\phi = 0$; $\lambda = 0$.

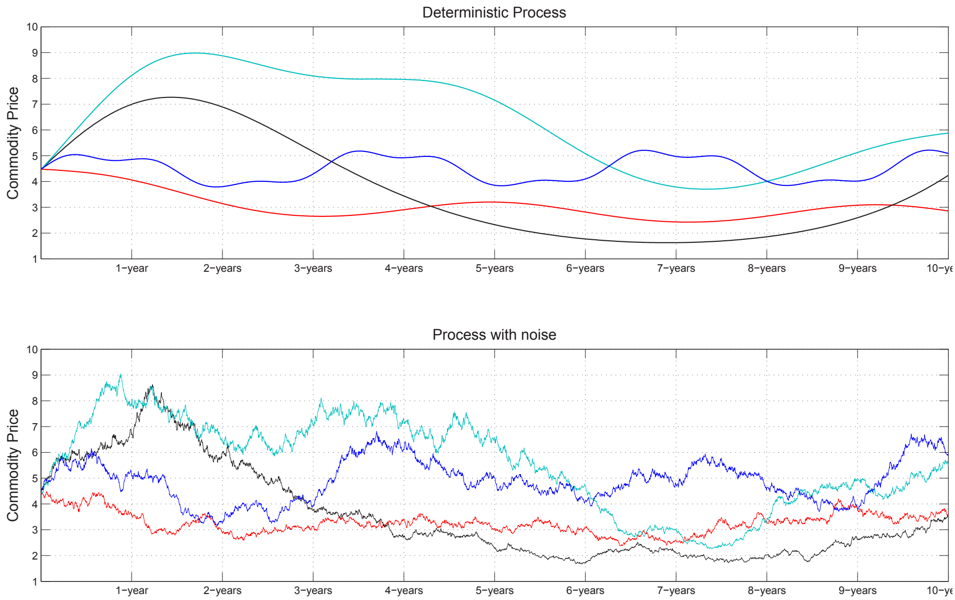
Figure 12: Term structure of interest rates for different values of the frequency and offset phase



The values of ω in the first graph are: *Blue Line*: = 0,05; *Lightblue Line*: = 0,1; *Black Line*: = 0,15; *Green Line*: = 0,2; and *Red Line*:= 0,5; and r_0 = 0,01; A_θ = 0,03; A_σ = 0,0002; κ = 0,1; ϕ = 0; λ = 0.

The values of ϕ in the second graph are: *Blue Line:* = 0, *Lightblue Line*: = $\frac{\pi}{6}$, *Black Line*: = $\frac{\pi}{4}$, *Green Line*: = $\frac{\pi}{2}$, and *Red Line*:= $\frac{3\pi}{4}$. And r_0 = 0,01, A_θ= 0,03, A_σ = 0,0002, κ = 0,1, ω = 0,2, λ = 0,

Figure 13: Spot price time series simulation for an arbitrary set of parameters



The first graph represents the drift process, that is setting $\sigma = 0$, The second graph represents the whole process with $\sigma = 0{,}2$.

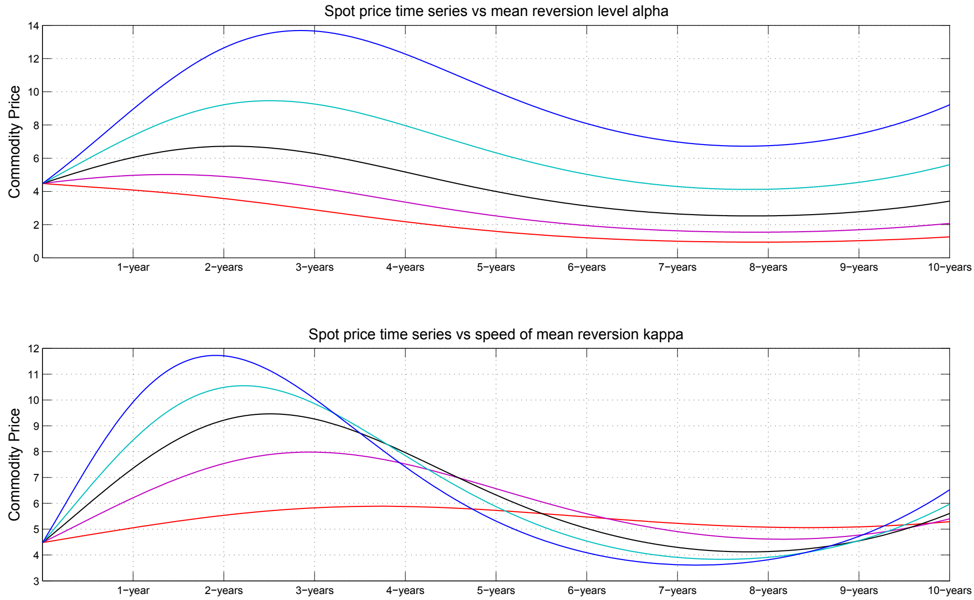
Red line: $\tilde{\alpha} = 1$; $\kappa = 0{,}5$; $A_{n=1,x} = 0{,}4$; $A_{n=1,y} = 0$; $A_{n=3,x} = 0$; $A_{n=3,y} = 0$; $\omega = 1{,}5$.

Black line: $\tilde{\alpha} = 2$; $\kappa = 0{,}5$; $A_{n=1,x} = 1$; $A_{n=1,y} = \frac{\pi}{2}$, $A_{n=3,x} = 0$; $A_{n=3,y} = 0$; $\omega = 0{,}4$.

Lightblue line: $\tilde{\alpha} = 2$; $\kappa = 0{,}5$; $A_{n=1,x} = 0{,}8$; $A_{n=1,y} = 0$; $A_{n=3,x} = 0{,}4$; $A_{n=3,y} = 0$; $\omega = 0{,}5$.

Blue line: $\tilde{\alpha} = 1{,}5$; $\kappa = 0{,}5$; $A_{n=1,x} = 0{,}6$; $A_{n=1,y} = 0$; $A_{n=3,x} = 0{,}5$; $A_{n=3,y} = 0$; $\omega = 2$.

Figure 14: Spot price time series simulation for an arbitrary set of parameters and no diffusion process, $\sigma = 0$



For both graphs: $A_{n=1,x} = 0{,}8$; $A_{n=1,y} = 0$; $n = 1$; $\omega = 0{,}5$.

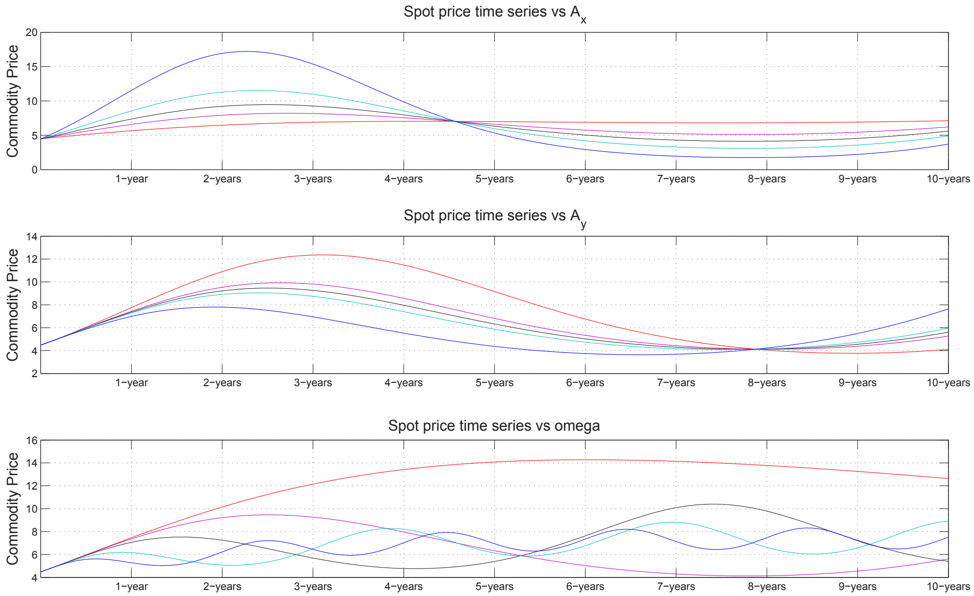
The first graph represents the spot price time series for $\kappa = 0{,}5$ and different values of $\tilde{\alpha}$:

Red line: $\tilde{\alpha} = 0{,}5$; Violet line: $\tilde{\alpha} = 1$, *Black line*: $\tilde{\alpha} = 1{,}5$; *Lightblue line*: $\tilde{\alpha} = 2$; *Blue line*: $\tilde{\alpha} = 2.5$.

The second graph represents the spot price time series for $\tilde{\alpha} = 2$ and different values of κ:

Red line: $\kappa = 0{,}1$; *Violet line*: $\kappa = 0{,}3$; *Black line*: $\kappa = 0{,}5$; *Lightblue line*: $\kappa = 0{,}7$; *Blue line*: $\kappa = 1$.

Figure 15: Spot price time series simulation for an arbitrary set of parameters and no diffusion process, σ = 0



For the three graphs: $\alpha = 2$; $\kappa = 0{,}5$; $n = 1$; $\sigma = 0$.

The first graph represents the spot price time series for $A_{n=1,y} = 0$; $\omega = 0{,}5$ and different values of $A_{n=1,x}$:

Red line: $A_{n=1,x} = 0{,}1$; *Violet line*: $A_{n=1,x} = 0{,}5$; *Black line*: $A_{n=1,x} = 0{,}8$; *Lightblue line*: $A_{n=1,x} = 1{,}2$; *Blue line*: $A_{n=1,x} = 2$.

The second graph represents the spot price time series for $A_{n=1,x} = 0{,}8$; $\omega = 0{,}5$ and different values of $A_{n=1,y}$:

Red line: $A_{n=1,y} = -0{,}5$; *Violet line*: $A_{n=1,y} = -0{,}1$; *Black line*: $A_{n=1,y} = 0$; *Lightblue line:* $A_{n=1,y} = 0{,}1$; *Blue line*: $A_{n=1,y} = 0{,}5$.

The third graph represents the spot price time series for $A_{n=1,x} = 0{,}8$, $A_{n=1,y} = 0$ and different values of ω:

Red line: $\omega = 0{,}1$; *Violet line:* $\omega = 0{,}5$; *Black line*: $\omega = 1$; *Lightblue line*: $\omega = 2$; *Blue line*: $\omega = \pi$.

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This thesis studies the stochastic behaviour of interest rates and commodity prices, extending the existing literature by allowing the underlying state variable to capture any possible seasonal or cyclical behaviour. In the first chapter, we propose a new model for the term structure of interest rates assuming that the instantaneous spot rate converges to a cyclical long-term level characterized by a Fourier series. Under this framework, we derive analytical expressions for the valuation of bonds and several interest rate derivative assets. The second chapter introduces a new square-root model for the yield curve where both the mean reversion level and the volatility are described by a harmonic oscillator. This model specification incorporates a good deal of flexibility preserving the analytical tractability. In the final chapter, we present a model for the logarithm of the commodity spot price with a reversion to a time dependent long-run level described by a Fourier series, obtaining closed-form expressions for a wide range of derivatives and study the fitting performance to market data.

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